

INDUCED SATURATION NUMBER

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ABSTRACT. In this paper, we discuss a generalization of the notion of saturation in graphs in order to deal with induced structures. In particular, we define $\text{indsat}(n, H)$, which is the fewest number of gray edges in a trigraph so that no realization of that trigraph has an induced copy of H , but changing any white or black edge to gray results in some realization that does have an induced copy of H .

We give some general and basic results and then prove that $\text{indsat}(n, P_4) = \lceil (n+1)/3 \rceil$ for $n \geq 4$ where P_4 is the path on 4 vertices. We also show how induced saturation in this setting extends to a natural notion of saturation in the context of general Boolean formulas.

1. INTRODUCTION

1.1. Saturation in graphs. A graph G is H -saturated if it does not contain H as a subgraph, but H occurs whenever any new edge is added to G . The Turán type problems deal with H -saturated graphs, in particular $\text{ex}(n; H) = \max\{|E(G)| : |V(G)| = n, G \text{ is } H\text{-saturated}\}$. Although previous work has been done in the field, $\text{sat}(n, H) = \min\{|E(G)| : |V(G)| = n, G \text{ is } H\text{-saturated}\}$ was first formally defined by Kászonyi and Tuza in [7]. They also find the saturation number for paths, stars, and matchings. In particular they find

$$\begin{aligned} \text{sat}(n, P_3) &= \left\lfloor \frac{n}{2} \right\rfloor, & \text{sat}(n, P_4) &= \begin{cases} k, & \text{if } n = 2k; \\ k+1, & \text{if } n = 2k-1, \end{cases} & \text{sat}(n, P_5) &= n - \left(\left\lfloor \frac{n-2}{6} \right\rfloor + 1 \right), \\ \text{sat}(n, P_h) &= n - \left\lfloor \frac{n}{3 \cdot 2^{h-1} - 1} \right\rfloor, \text{ when } h = 2k \geq 6, & \text{and } \text{sat}(n, P_h) &= n - \left\lfloor \frac{n}{2^{k+1} - 2} \right\rfloor, \text{ when } h = 2k+1 \geq 7. \end{aligned}$$

They also find a general upper bound for saturation number. That is for any graph H there exists a constant $c = c(H)$, such that $\text{sat}(n, H) < cn$.

Pre-dating [7], Erdős, Hajnal, and Moon in 1964 found the saturation number of complete graphs in [4]; that is, $\text{sat}(n, K_h) = (h-2)n - \binom{h-1}{2}$. Eight years later in [10], Ollmann found the saturation number of the four cycle to be $\text{sat}(n, C_4) = \lceil 3(n-5)/2 \rceil$. More recently, R. Faudree, Ferrara, Gould and Jacobson [6] found the saturation number of tK_p and Chen [2] found the saturation number of C_5 . A more complete background of known saturation results is provided in the dynamic survey by J. Faudree, R. Faudree and Schmitt [5].

1.2. Notation and definitions. We would like to generalize the notion of saturation to include induced subgraphs. To do this we use definitions given by Chudnovsky [3]. Let a *trigraph*¹ T be a quadruple $(V(T), EB(T), EW(T), EG(T))$, with $V(T)$ being the vertices of the trigraph, $EB(T)$ being the black edges, $EW(T)$ being the white edges, and $EG(T)$ being the gray edges. We will think of these as edges, non-edges, and ‘free’ edges, where ‘free’ means those edges can either be black or white. We note here that if $EG(T) = \emptyset$, then our trigraph is just a graph.

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¹Doug West has suggested the term “fuzzy graphs” to comport with the notion of fuzzy sets. However, Rosenfeld [11] has used this term for a different object and so we choose to use Chudnovsky’s notation.

Definition 1.1. A *realization* of a trigraph T is a graph G with $V(G) = V(T)$ and $E(G) = EB(T) \cup S$ for some subset S of $EG(T)$. That is, we set some gray edges to black edges and the remaining gray edges to white.

For any graph H , we say that a trigraph T **has a realization of H** if there is a realization of T which has H as an induced subgraph.

A trigraph T is *H -induced-saturated* if no realization of T contains H as an induced subgraph, but H occurs as an induced subgraph of some realization whenever any black or white edge of G is changed to gray. The **induced saturation number** of H with respect to n is defined to be $\text{indsat}(n, H) = \min\{|EG(T)| : |V(T)| = n, T \text{ is } H\text{-induced-saturated}\}$.

Remark 1.2. By definition, the only trigraphs on fewer than $|V(H)|$ vertices that are H -induced-saturated are those in which all edges are gray.

In colloquial terms, to find $\text{indsat}(n, H)$ is to find a trigraph T on n vertices with the fewest number of gray edges having the property that there is no induced subgraph H but if any black or white edge is changed to gray, then we find H as an induced subgraph.

If an edge has endvertices v and w , we denote it vw or wv . For a vertex v in a trigraph T , the *white neighborhood* of v is the set $\{w : vw \in EW(T)\}$. The *black neighborhood* and *gray neighborhood* are defined similarly. In addition, we write $v \sim w$ to mean that $vw \in EB(T)$ and $v \not\sim w$ to mean that $vw \in EW(T)$. In a graph, we say that vertices $v_1 v_2 \dots v_k$ *form a path* if $v_i v_{i+1}$ is an edge for $i = 1, \dots, k-1$ but all other pairs of these vertices are nonedges.

Since induced saturation involves taking a trigraph T and changing an otherwise nongray edge e to gray, we denote $T \vdash e$ to be the trigraph that results from making e a gray edge. This corresponds with the “+” notation in ordinary saturation.

The *complement*, \bar{T} of a trigraph T is a trigraph with $V(\bar{T}) = V(T)$, $EB(\bar{T}) = EW(T)$, $EG(\bar{T}) = EG(T)$, and $EW(\bar{T}) = EB(T)$, which extends the definition of graph complement. We define a *gray component* to be a set of vertices that are connected in the graph $(V(T), EG(T))$. We do not require that every edge within the component is gray, just that there is a gray path connecting any pair of vertices. Analogously, we define *white*, *black*, *black/gray*, and *white/gray components*.

If V_1 and V_2 are disjoint sets in $V(T)$, then $T[V_1, V_2]$ denotes the set of edges with one endpoint in V_1 and the other in V_2 and $T[V_1]$ denotes the subtrigraph induced by V_1 .

1.3. Main Result. In Section 2, we establish a few very basic results regarding induced saturation, such as the fact that $\text{indsat}(n, H)$ is bounded by $\text{sat}(n, H)$, but the main result is the induced saturation number for paths of length 4.

Theorem 1.3. $\text{indsat}(n, P_4) = \lceil \frac{n+1}{3} \rceil$, for all $n \geq 4$.

The proof of this result comprises the rest of the paper. Section 3 gives the upper bound $\text{indsat}(n, P_4) \leq \lceil \frac{n+1}{3} \rceil$. Section 4 gives the lower bound $\text{indsat}(n, P_4) \geq \lceil \frac{n+1}{3} \rceil$. Section 4.1 gives a number of general facts regarding P_4 -induced-saturated trigraphs which are used repeatedly in the subsequent sections. Section 4.2 is the main proof of the lower bound. Section 4.2.1 contains the proof that a P_4 -induced-saturated trigraph contains at least one gray edge and Section 4.2.2 contains the proof of an important technical lemma. Section 5 contains the proofs of the facts enumerated in Section 4.1. Section 6 contains some concluding remarks.

1.4. Generalizations and Applications. Although we are concerned with induced graph saturation, there is a more general view, in terms of satisfiability. Given a disjunctive normal form (DNF), we want to find a partial assignment of variables such that (1) there is no way to complete the assignment to a true one but (2) if any one of the assigned variables were unassigned, then this new partial assignment can be completed to a true one.

In the specific case of graphs, the DNF constructed from the set of pairs of n vertices is one comprised of clauses, each of which represents an instance of a potential induced copy of H . For instance, if e_1, \dots, e_6

represent unordered pairs such that e_1, e_2, e_3 being edges and e_4, e_5, e_6 being nonedges induces P_4 , then the corresponding clause is $x_1 \wedge x_2 \wedge x_3 \wedge \overline{x_4} \wedge \overline{x_5} \wedge \overline{x_6}$, where the variable x_i corresponds to the pair e_i , for $i = 1, \dots, 6$.

The trigraph has a number of applications related to Szemerédi's Regularity Lemma [12] (see also [8, 9]). A trigraph can also be thought of as a reduced graph in which the a black edge represents a pair with density close to 1, a white edge represents a pair with density close to 0 and a gray edge represents a pair with density neither near 1 nor near 0. Such a configuration is used in a number of applications of the regularity lemma related to induced subgraphs. See, for instance, [1].

2. BASIC RESULTS

In this section we establish some basic results on the induced saturation number. We find a relationship between induced saturation and classical saturation as well as establish the induced saturation number for a few families of graphs.

The first result presented relates the saturation number to the induced saturation number.

Theorem 2.1. $\text{indsat}(n, H) \leq \text{sat}(n, H)$ for all graphs H and all positive integers $n \geq |V(H)|$.

Proof. We can take the graph that gives the upper bound for saturation number and change all the edges to gray edges and leave all non-edges white. Then clearly changing any white edge to gray forms an induced graph H in the same way as adding any black edge created an H in the case of saturation of graphs. \square

For the complete graph, the numbers are identical:

Theorem 2.2. If K_h is an h -vertex complete graph, then $\text{indsat}(n, K_h) = \text{sat}(n, K_h) = (h-2)n - \binom{h-1}{2}$ for all $n \geq h \geq 3$.

Proof. This is a direct result of the fact that K_h being an induced subgraph is identical to K_h being a subgraph. All K_h -induced-saturated trigraphs have no black edges. \square

One may wonder if it is ever the case that $\text{indsat}(n, H) < \text{sat}(n, H)$. That is, can we improve on the bound provided by the saturation number? Theorem 2.3 shows that the answer is yes. We know from Kászonyi and Tuza in [7] that $\text{sat}(n, P_3) = \lfloor \frac{n}{2} \rfloor$ and, trivially, $\text{sat}(n, K_h) \geq \lfloor \frac{n}{2} \rfloor$, for $h \geq 4$.

Theorem 2.3. Let K_h^- denote a graph on h vertices with exactly one nonedge. Then, $\text{indsat}(n, K_h^-) = 0$ for all $n \geq h$.

Proof. For an upper bound, we choose our trigraph T to be the complete graph of black edges on n vertices. Clearly, T does not contain an induced K_h^- . Now, choose an arbitrary edge, e , in T , and consider $T \div e$. Since all edges were black, the new gray edge can be used as the nonedge in a copy of K_h^- . \square

Since P_3 , the path on 3 vertices, is the same as K_3^- , $\text{indsat}(n, P_3) = 0$ as well.

Corollary 2.4. $\text{indsat}(n, P_3) = 0$, for all $n \geq 3$.

So, not only can the induced saturation number differ drastically from K_h to K_h^- , but it can be much less than the saturation number for the same graph. Theorem 1.3 establishes that the induced saturation number $\text{indsat}(n, P_4)$ is both nonzero for $n \geq 2$ and is strictly less than $\text{sat}(n, P_4)$ for $n = 5$ and $n \geq 7$.

3. PROOF OF UPPER BOUND IN THE MAIN RESULT

In this section, we prove the following via constructions.

Lemma 3.1. $\text{indsat}(n, P_4) \leq \lceil \frac{n+1}{3} \rceil$ for $n \geq 4$.

Proof. We have three constructions depending on the remainder of n upon division by 3. We will denote the trigraph construction on n vertices by T'_n .

Construction 3.2. *The trigraph T'_n for $n \geq 4$ has the following vertex set:*

$$V(T'_n) = \begin{cases} \bigcup_{i=1}^k \{a_i, b_i, c_i\} \cup \{a_{k+1}, b_{k+1}\}, & \text{if } n = 3k + 2; \\ \bigcup_{i=1}^k \{a_i, b_i, c_i\} \cup \{a_{k+1}, b_{k+1}\} \cup \{c_0\}, & \text{if } n = 3k + 3; \\ \bigcup_{i=1}^k \{a_i, b_i, c_i\} \cup \{a_{k+1}, b_{k+1}\} \cup \{a_0, b_0\}, & \text{if } n = 3k + 4; \end{cases}$$

The edges are colored as follows:

$$\begin{aligned} a_i b_i &\in EG(T'_n), \quad \forall i; \\ c_0 c_1 &\in EG(T'_n), \quad \text{if } n \equiv 0 \pmod{3}; \\ a_i a_j, a_i b_j, b_i b_j &\in EW(T'_n), \quad \forall i \neq j; \\ c_i c_j &\in EB(T'_n), \quad \forall i \neq j, \{i, j\} \neq \{0, 1\}; \\ c_i a_j, c_i b_j &\in EB(T'_n), \quad \forall i \leq j; \\ c_i a_j, c_i b_j &\in EW(T'_n), \quad \forall i > j. \end{aligned}$$

Remark 3.3. *The subtrigraph of T'_n induced by $\bigcup_{i=1}^k \{a_i, b_i, c_i\}$ resembles the so-called half-graph. In fact $\bigcup_{i=1}^k \{a_i, c_i\}$ is the half-graph on $2k$ vertices if black edges are interpreted as the edges of the graph and white edges are interpreted as the nonedges.*

The various constructions are shown in Figures 1, 2 and 3. It is easy to compute that, in each case, there are exactly $\lceil \frac{n+1}{3} \rceil$ gray edges in each T'_n for $n \geq 4$.

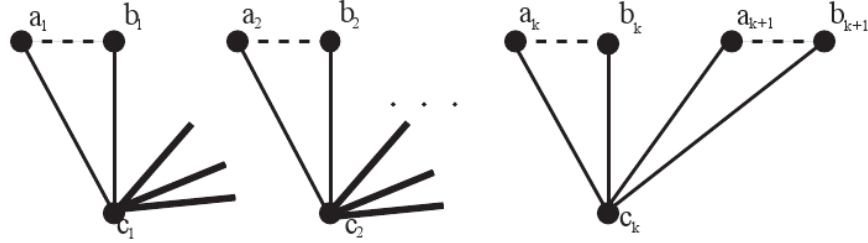


FIGURE 1. T'_n when $n = 3k + 2$

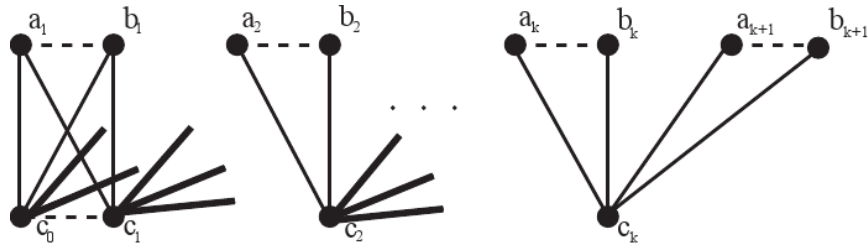
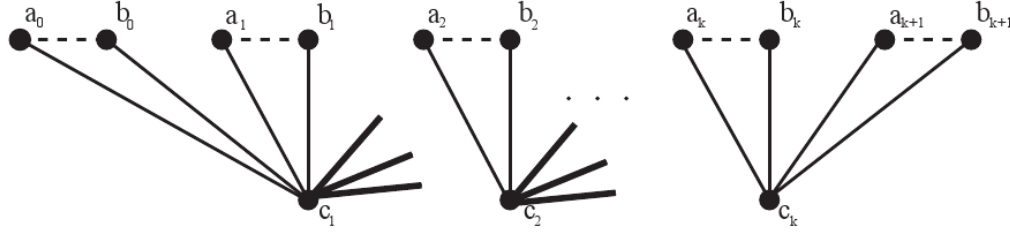


FIGURE 2. T'_n when $n = 3k + 3$

It remains to show that, for $n \geq 4$, the graph T'_n is P_4 -induced-saturated. Recall to do this, first we must show that T does not contain a realization which has a P_4 as an induced subgraph. Second, we must show that if we flip any white or black edge to gray, then T does contain a realization which has a P_4 as an induced subgraph. We leave it to the reader to verify this in the case of $n = 4$, in which T'_4 is two disjoint gray edges and a white 4-cycle. So, we may assume $n \geq 5$, i.e., $k \geq 1$.

FIGURE 3. T'_n when $n = 3k + 4$

First, we show that T'_n does not contain an induced P_4 . If $n = 3k + 3$, then if c_0 is in such a realization, then so must c_1 because $c_0x \in EB(T'_n)$ if $x \neq c_1$. But $\{c_0, c_1, x, y\}$ must contain a black C_4 for all distinct $x, y \notin \{c_0, c_1\}$, hence no P_4 realization is possible. So, c_0 is in no realization of a P_4 . In addition, no realization of P_4 can have 3 or 4 vertices from $\{c_1, \dots, c_k\}$ because they induce a black C_3 . If we have four distinct vertices $\{c_i, c_j, x, y\}$, where $1 \leq i < j \leq k$, then there is either a black C_3 or a white C_4 , hence no realization of P_4 . If we have no vertices from $\{c_1, \dots, c_k\}$, then we have a white C_4 and so no realization of P_4 . Finally, consider four distinct vertices $\{c_i, x, y, z\}$ where $x, y, z \notin \{c_1, \dots, c_k\}$. We may assume that two of $\{x, y, z\}$ are adjacent via a gray edge, otherwise there is a white C_3 . Without loss of generality, the vertices are of the form $\{c_i, a_j, b_j, a_\ell\}$. Regardless of the values of j and ℓ relative to i , there is either a black star with 3 leaves, a white star with 3 leaves or a white C_4 , all of which preclude a realization of P_4 .

Second, we show that if any black or white edge in T'_n is flipped to gray then the new trigraph does contain an induced P_4 . We will take care of this with case analysis, focusing primarily on Figure 1 first and then filling in the missing edges for the other two constructions. One observation we will make is that if trigraph T contains no realization of P_4 but $T \div e$ does, for some white edge e , then the realization must use e as a black edge. The complementary observation (flipping a black edge requires it be used as a white edge) holds as well.

- (1) **Flip $a_i a_j$:** We consider $T'_n \div a_i a_j$ for $i < j$, which flips $a_i a_j$ from white to gray. In this case we have an induced P_4 with vertices $b_i a_i a_j b_j$. By symmetry, we conclude that $T'_n \div a_i b_j$ and $T'_n \div b_i b_j$ have a realization of P_4 , for all $i \neq j$.
- (2) **Flip $c_i c_j$:** We consider $T'_n \div c_i c_j$ for $i < j$ and $\{i, j\} \neq \{0, 1\}$, which flips $c_i c_j$ from black to gray. In this case we have an induced P_4 with vertices $a_i c_i a_j c_j$.
- (3) **Flip $c_i a_i$:** We consider $T'_n \div c_i a_i$, which flips $c_i a_i$ from black to gray. In this case we have an induced P_4 with vertices $a_i b_i c_i a_{i+1}$. By symmetry, we conclude that $T'_n \div c_i b_i$ has a realization of P_4 , for all i .
- (4) **Flip $c_i a_j, i < j$:** We consider $T'_n \div c_i a_j$ for $i < j$, which flips $c_i a_j$ from black to gray. In this case we have an induced P_4 with vertices $a_i c_i b_j a_j$. By symmetry, we conclude that $T'_n \div c_i b_j$ has a realization of P_4 , for all $i < j$.
- (5) **Flip $c_i a_j, i > j$:** We consider $T'_n \div c_i a_j$ for $i > j$, which flips $c_i a_j$ from white to gray. In this case we have an induced P_4 with vertices $b_j a_j c_i a_i$. By symmetry, we conclude that $T'_n \div c_i b_j$ has a realization of P_4 , for all $i > j$.

Therefore, T'_n is P_4 -induced-saturated for all $n \geq 5$ and, together with the case $n = 4$, we are done. \square

4. PROOF OF LOWER BOUND IN THE MAIN RESULT

In this section, we prove the lower bound found in Theorem 1.3:

Theorem 4.1. $\text{indsat}(n, P_4) \geq \lceil \frac{n+1}{3} \rceil$.

The proof is done by strong induction and bulk of the work will be done by the technical lemma, Lemma 4.11. Before the main proof, some basic facts will be enumerated in Section 4.1. Recall that a trigraph T that is P_4 -induced-saturated with $|V(T)| \leq 3$ is a complete gray graph. To ease the statements of the proof, we state a few definitions:

Let T be a trigraph. A star in some color class of T is a *trivial star* if it has two (or fewer) vertices. If a star has three or more vertices, it is called a *nontrivial star*. A connected component of a color class of T is a *trivial component* if it consists of a single vertex. If it consists of at least two vertices, it is called a *nontrivial component*.

4.1. General Facts. The proofs of the following facts will appear in Section 5. We state them here as they provide a feel for how the proof of the main result will be handled.

Fact 4.2. *If a trigraph T is P_4 -induced-saturated, then the complement of T , \overline{T} , is also P_4 -induced-saturated.*

Fact 4.3. *Let T be a trigraph which is P_4 -induced-saturated. If $V(T)$ is partitioned into V_1 and V_2 such that either each edge in $T[V_1, V_2]$ is white or each edge in $T[V_1, V_2]$ is black, then both $T[V_1]$ and $T[V_2]$ are P_4 -induced-saturated.*

Let T be a trigraph with distinct vertices v_1 and v_2 and $S \subseteq V(T) - \{v_1, v_2\}$. We say v_1 and v_2 have the same neighborhood in S if v_1s and v_2s have the same color for every $s \in S$.

Fact 4.4. *Let T be a trigraph that is P_4 -induced-saturated. Let S be a subset of $V = V(T)$ such that $T[S, V - S]$ has no gray edges. If every pair of vertices in S has the same neighborhood in $V - S$, then S is P_4 -induced-saturated.*

By Fact 4.5, the gray components of size at least 2 in P_4 -induced-saturated trigraphs are either K_3 's or S_k 's for $k \geq 2$.

Fact 4.5. *If T is a P_4 -induced-saturated trigraph on $n \geq 2$ vertices, then each nontrivial gray component in T is either a complete gray K_3 or a gray star, S_k , $k \geq 2$, where S_k denotes the star on k vertices.*

Facts 4.6 and 4.7 establish a partition of $V(T)$ according to how those vertices relate to a particular gray component. Note that Fact 4.6 also applies if the gray component is an edge on 2 vertices.

Fact 4.6. *Let T be a trigraph that is P_4 -induced-saturated. If $k \geq 2$ and $\{u, v_1, v_2, \dots, v_{k-1}\}$ is a gray star in T with center u , then $V(T) - \{u, v_1, v_2, \dots, v_{k-1}\}$ can be partitioned into X , Y and Z such that the following occur:*

- The edges xv_i and xv_i are black for all $i \in \{1, \dots, k-1\}$ and for all $x \in X$.
- The edges yu and yv_i are white for all $i \in \{1, \dots, k-1\}$ and for all $y \in Y$.
- One of the following occurs:
 - The edges $v_i v_j$ and $v_i z$ are white and uz are black for all distinct $i, j \in \{1, \dots, k-1\}$ and for all $z \in Z$.
 - The edges $v_i v_j$ and $v_i z$ are black and uz are white for all distinct $i, j \in \{1, \dots, k-1\}$ and for all $z \in Z$.

Note that in the case where $k = 2$, Fact 4.6 still holds, but the subcases for the behavior of vertices in Z are equivalent, depending on the labeling of the vertices. Regardless, if uv is a gray component, we may assume that the edges in $T[Z, \{u\}]$ are black and the edges in $T[Z, \{v\}]$ are white.

Fact 4.7. *Let T be a trigraph that is P_4 -induced-saturated. If $\{v_1, v_2, v_3\}$ is a gray triangle in T , then $V(T) - \{v_1, v_2, v_3\}$ can be partitioned into X and Y such that the following occur:*

- The edges xv_i are black for all $i \in \{1, 2, 3\}$ and for all $x \in X$.
- The edges yv_i are white for all $i \in \{1, 2, 3\}$ and for all $y \in Y$.

4.2. Proof of Theorem 4.1. We will prove Theorem 4.1 by strong induction on n . The inductive hypothesis is: If T is a P_4 -induced-saturated trigraph on $n \geq 2$ vertices, then $|EG(T)| \geq \lceil \frac{n+1}{3} \rceil$.

Note that the statement is true for the trivial base cases of $n = 2, 3$. So, let T be a trigraph on $n \geq 4$ vertices that is P_4 -induced-saturated. First, we use Lemma 4.8, proven in Section 5 to establish that there must be at least one gray edge.

Lemma 4.8. *If T is a P_4 -induced-saturated trigraph on $n \geq 2$ vertices, then T has a gray edge.*

By Fact 4.5, we know that gray edges must be in components that are either triangles or stars and by Facts 4.6 and 4.7, each nontrivial gray component C partitions $V(T)$ into sets C , X , Y and Z , any of which could be empty.

Claim 4.9. *With X, Y, Z defined as above, the edges in $T[Z, X]$ are black and the edges in $T[Z, Y]$ are white.*

Proof of Claim 4.9. If C is a gray triangle or Z is otherwise empty, the claim is vacuous, so we assume that C is a star and Z is not empty. Without loss of generality, let us assume uv is a gray edge in the star such that the edges in $T[Z, \{u\}]$ are black and the edges in $T[Z, \{v\}]$ are white. Let $z \in Z$. If $x \in X$ and zx is white, then $zuxv$ has a realization of P_4 . If $y \in Y$ and zy is black, then $yzuv$ has a realization of P_4 . This proves Claim 4.9. \square

Let C_0 be a gray component in which Z is maximum-sized.

Claim 4.10. *With X, Y, Z defined by C_0 , there are no gray edges in $T[X, Y]$.*

Proof of Claim 4.10. Let xy be a gray edge such that $x \in X$ and $y \in Y$. By Claim 4.9, the set $S = Z \cup \{u, v\}$ has the property that the edges in $T[\{x\}, Z]$ are black and the edges in $T[\{y\}, Z]$ are white. Hence, the gray component that xy is in has a larger Z -set than that formed by C_0 . \square

With C_0 , X , Y and Z defined as above, we can turn to Lemma 4.11:

Lemma 4.11. *Let T be a P_4 -induced-saturated trigraph and let X, Y be disjoint subsets of $V = V(T)$, $X \cup Y \neq \emptyset$ such that the following is true:*

- (1) $T[X, Y]$ has no gray edge.
- (2) Each edge in $T[X, V - (X \cup Y)]$ is black and each edge in $T[Y, V - (X \cup Y)]$ is white.
- (3) There exist vertices $u, v \in V(T) - (X \cup Y)$ such that uv is a gray edge.

If $|X| + |Y| \geq 2$, then the number of gray edges in $T[X \cup Y]$ is at least $\lceil \frac{|X| + |Y|}{3} \rceil$.

To see that we can apply Lemma 4.11 to finish the proof, note the following: Condition (1) holds by Claim 4.10. Condition (2) holds because of the definition of X and Y from Facts 4.6 and 4.7 as well as because of Claim 4.10. Condition (3) holds because of the existence of C_0 .

Let uv be a gray edge in C_0 such that the edges in $T[Z, \{u\}]$ are black and the edges in $T[Z, \{v\}]$ are white.

Case (a.) $|Z| = 1$:

We will show that this case cannot occur. Suppose it does and let $Z = \{z\}$. Consider a realization of P_4 , call it P , in the trigraph $T \div z$. By the definition of P , it must use zv as a black edge. Then P cannot contain an $x \in X$ because if it did, both xz and xv would be black, creating a black triangle and preventing P from being a realization of P_4 . But, P cannot contain a member of Y either, because there is no black/gray path from a vertex in Y to either z or v that avoids X .

Since P cannot contain a vertex in $X \cup Y$, it must be contained in the vertices $\{z, u, v\}$, but there are only three of those, a contradiction. So, Case (a.) cannot occur.

Case (b.) $Z = \emptyset$, $|X| + |Y| \leq 1$:

In this case, $|V(T)| \leq |C_0| + 1$. If T has at least 4 vertices, then C_0 has at least $|C_0| - 1 = n - 2$ gray edges,

because C_0 is a gray component. Since $n - 2 \geq \lceil (n + 1)/3 \rceil$ for all $n \geq 4$, Case (b.) satisfies the conditions of the theorem.

Case (c.) $Z = \emptyset$, $|X| + |Y| \geq 2$:

We can apply Lemma 4.11. The number of gray edges in T is at least

$$\left\lceil \frac{|X| + |Y|}{3} \right\rceil + |C_0| - 1 = \left\lceil \frac{n + 2|C_0| - 3}{3} \right\rceil \geq \left\lceil \frac{n + 1}{3} \right\rceil.$$

Thus, Case (c.) satisfies the conditions of the theorem.

Case (d.) $|Z| \geq 2$, $|X| + |Y| \leq 1$:

By Fact 4.4 (where $S = Z$) the subtrigraph $T[Z]$ is P_4 -induced-saturated. By the inductive hypothesis, the number of gray edges in $T[Z]$ is at least $\left\lceil \frac{|Z| + 1}{3} \right\rceil$. Hence, the number of gray edges in T is at least

$$\left\lceil \frac{|Z| + 1}{3} \right\rceil + |C_0| - 1 = \left\lceil \frac{|Z| + 3|C_0| - 2}{3} \right\rceil \geq \left\lceil \frac{(n - |C_0| - 1) + 3|C_0| - 2}{3} \right\rceil = \left\lceil \frac{n + 2|C_0| - 3}{3} \right\rceil \geq \left\lceil \frac{n + 1}{3} \right\rceil.$$

Thus, Case (d.) satisfies the conditions of the theorem.

Case (e.) $|Z| \geq 2$, $|X| + |Y| \geq 2$:

Again, by Fact 4.4 the subtrigraph $T[Z]$ is P_4 -induced-saturated. We may apply both the inductive hypothesis and Lemma 4.11. The number of gray edges in T is at least

$$\left\lceil \frac{|Z| + 1}{3} \right\rceil + \left\lceil \frac{|X| + |Y|}{3} \right\rceil + |C_0| - 1 = \left\lceil \frac{|X| + |Y| + |Z| + 1 + 3|C_0| - 3}{3} \right\rceil = \left\lceil \frac{n + 2|C_0| - 2}{3} \right\rceil \geq \left\lceil \frac{n + 2}{3} \right\rceil.$$

Thus, Case (e.) satisfies the conditions of the theorem. So, the proof of Theorem 4.1 is complete. \square

4.2.1. Proof of Lemma 4.8. We will prove that every P_4 -induced-saturated trigraph on $n \geq 2$ vertices has at least one gray edge by strong induction on n . The base cases of $n = 2, 3$ are true by definition.

Let $n \geq 4$. For the sake of contradiction, assume T is a P_4 -induced-saturated trigraph on vertex set V , $|V| \geq 4$, which does not contain a gray edge. Since T is assumed to have no gray edges, we may regard it as a graph and will sometimes use the language of edges and nonedges.

There must be an induced P_3 in T , otherwise T is disjoint cliques. If T were disjoint cliques, then Fact 4.3 implies that each of the cliques is P_4 -induced-saturated. We may apply the inductive hypothesis unless each clique is of size 1 or there is one clique of size n . It is trivial to see that neither case is possible because a trigraph on at least 4 vertices with all white edges or with all black edges cannot be P_4 -induced-saturated.

Let the induced P_3 in T be uxv . Let the set X be the vertices that are adjacent to both u and v . The set Z_1 will be the vertices that are adjacent to u but not v . Likewise, the set Z_2 will be the vertices that are adjacent to v but not u . Finally, let the set Y be all vertices that are not adjacent to u or v . See Figure 4.

First, note that $x \in X$, hence $X \neq \emptyset$.

Second, we show that all edges in $T[X, Z_1]$ and in $T[X, Z_2]$ must be black. To see this, let $x \in X$, $z_1 \in Z_1$, $z_2 \in Z_2$. Without loss of generality, assume $xz_1 \notin EB(T)$. In this case, T contains an induced P_4 , namely z_1uxv , contradicting that T is P_4 -induced-saturated. By a symmetric argument, we may assume $xz_2 \in EB(T)$.

We will partition the vertices of X into equivalence classes according to their neighborhoods in Y . For each $x \in X$, denote $N_Y^B(x)$ to be the set of $y \in Y$ such that the edge xy is black. In order to proceed, we need two claims. Claim 4.12 establishes that the equivalence classes have the same color edge between them.

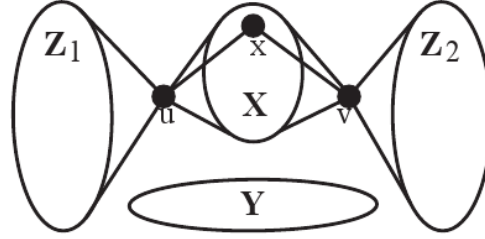


FIGURE 4. General set-up for Proof of Lemma 4.8. Trigraph T containing a P_3 with vertices uxv .

Claim 4.12. *If $x, x' \in X$ have different neighborhoods in Y , then xx' is black.*

If $y, y' \in Y$ have different neighborhoods in X , then yy' is white.

Proof of Claim 4.12. Suppose xx' is not black. If $\tilde{y} \in Y$ where $x\tilde{y}$ is black and $x'\tilde{y}$ is white, then $\tilde{y}xux'$ is a realization of P_4 .

Suppose yy' is not white. If $\tilde{x} \in X$ where $y\tilde{x}$ is black and $y'\tilde{x}$ is white, then $u\tilde{x}yy'$ is a realization of P_4 . This proves Claim 4.12. \square

Claim 4.13 establishes that the black neighborhoods in Y nest.

Claim 4.13. *For all $x, x' \in X$, either $N_Y^B(x) \subseteq N_Y^B(x')$ or $N_Y^B(x) \supseteq N_Y^B(x')$.*

For all $y, y' \in Y$, either $N_X^B(y) \subseteq N_X^B(y')$ or $N_X^B(y) \supseteq N_X^B(y')$.

Proof of Claim 4.13. Suppose $\tilde{y} \in N_Y^B(x) - N_Y^B(x')$ and $\tilde{y}' \in N_Y^B(x') - N_Y^B(x)$. Using Claim 4.12, xx' is black and $\tilde{y}\tilde{y}'$ is white. Hence, $\tilde{y}x\tilde{y}'$ is a realization of P_4 .

Suppose $\tilde{x} \in N_X^B(y) - N_X^B(y')$ and $\tilde{x}' \in N_X^B(y') - N_X^B(y)$. Using Claim 4.12, yy' is white and $\tilde{x}\tilde{x}'$ is black. Hence, $y\tilde{x}\tilde{x}'y'$ is a realization of P_4 . This proves Claim 4.13. \square

So, we define an equivalence relation on the vertices in X so that $x, x' \in X$ are equivalent if and only if $N_Y^B(x) = N_Y^B(x')$. It is easy to see that this is an equivalence relation. Let the equivalence classes be X_1, \dots, X_ℓ with the property that $x_i \in X_i$ and $x_{i+1} \in X_{i+1}$ imply $N_Y^B(x_i) \subset N_Y^B(x_{i+1})$, for $i = 1, \dots, \ell - 1$.

By definition, $X_\ell \neq \emptyset$ so let $x \in X_\ell$ and consider some $y \in Y$ such that $y \notin N_Y^B(x)$ (implying y has no neighbors in X). If $z_1 \in Z_1$ such that $yz_1 \in EB(T)$, then since xz_1 is black, we have an induced P_4 , namely vxz_1y , a contradiction. But, if y has no neighbors in Z_1 , then y and u can have no common neighbors. Therefore, $\text{dist}(y, u) \geq 3$.

If $\text{dist}(y, u) < \infty$, then there is an induced path on at least 4 vertices between y and u , giving the existence of an induced P_4 . If $\text{dist}(y, u) = \infty$, then either T is a trigraph with all white edges (hence not P_4 -induced-saturated) or T has at least one nontrivial component. By Fact 4.3, each component is P_4 -induced-saturated and so applying the inductive hypothesis to any nontrivial component gives a gray edge.

Finally, we may conclude that the edges in $T[X_\ell, V - X_\ell]$ are all black. By Fact 4.3, both X_ℓ and $V - X_\ell$ are P_4 -induced-saturated. Applying the inductive hypothesis to whichever of those sets is nontrivial gives a gray edge and a contradiction. This concludes the proof of Lemma 4.8. \square

4.2.2. Proof of Lemma 4.11. We wish to take note that this lemma requires the strong inductive hypothesis from the proof of Theorem 4.1.

If either X or Y is empty, then the other set has at least two vertices and the inductive hypothesis gives that there are at least $\left\lceil \frac{|X|+|Y|+1}{3} \right\rceil$ gray edges.

As in the proof of Lemma 4.8, we will partition the vertices of X into equivalence classes according to their neighborhoods in Y . The statements of Claim 4.12 and 4.13 were written to apply to this lemma as well, even though gray edges are permitted in $T[X]$ and $T[Y]$. So, the edges between equivalence classes of X are black and between equivalence classes of Y are white. Furthermore, the sets $N_Y^B(x)$ – the black neighborhood of x in set Y – form a nesting family.

So, again define an equivalence relation on the vertices in X so that $x, x' \in X$ are equivalent if and only if $N_Y^B(x) = N_Y^B(x')$. Let the equivalence classes be X_1, \dots, X_ℓ . This, in turn, defines a partition of Y . For $j = 1, \dots, \ell + 1$, the set Y_j has the property that the edges in $T[X_i, Y_j]$ are white for $i = 1, \dots, j - 1$ and the edges in $T[X_i, Y_j]$ are black for $i = j, \dots, \ell$. See Figure 5.

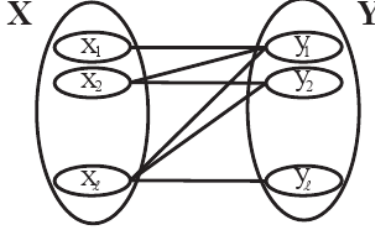


FIGURE 5. A decomposition of X and Y into equivalence classes. The pair (X_i, Y_j) consists of black edges if $i \geq j$; otherwise, it consists of white edges.

The sets Y_2, \dots, Y_ℓ are nonempty because if Y_j is empty, then $N_Y^B(x) = Y_1 \cup \dots \cup Y_{j-1}$ for all $x \in X_{j-1} \cup X_j$, contradicting the definition of the equivalence classes. By Fact 4.4, each of the sets $X_1, \dots, X_\ell, Y_1, \dots, Y_{\ell+1}$ is P_4 -induced-saturated. Thus, if the sets have size at least two, there will be enough gray edges to prove the lemma. We will ensure that not too many of the sets are of size one.

Claim 4.14. *For $i = 1, \dots, \ell$, it cannot be the case that both $|X_i| = 1$ and $|Y_i| = 1$ or that $|X_i| = 1$ and $|Y_{i+1}| = 1$.*

Proof of Claim 4.14. First, suppose that $|X_i| = |Y_i| = 1$ and let $X_i = \{x_i\}$ and $Y_i = \{y_i\}$. Consider $T \div x_i y_i$ and note we may now regard $x_i y_i$ as a white edge because it was black in T itself. There must be a black neighbor of y_i in the realization and that must be $x_j \in X_j$ for some $j > i$. (See the diagram in Figure 5.) But x_j is a black neighbor of x_i also. So, $x_i x_j y_i$ is a realization of P_3 . Suppose there is an additional vertex, w , in the realization of the P_4 . If $w \in X - X_j$, then $w x_i x_j$ induces a black triangle and if $w \in X_j$ then $w x_i x_j y_i$ has a black C_4 . Both of these cases produce a contradiction. If $w \in Y$, then $w y_i$ is white and being in the realization of P_4 requires $w x_i$ to be black. However, this means $w x_j$ is also black because $j > i$. This is also a contradiction. Hence there is no realization of P_4 in $T \div x_i y_i$.

Second, suppose that $|X_i| = |Y_{i+1}| = 1$ and let $X_i = \{x_i\}$ and $Y_{i+1} = \{y_{i+1}\}$. Consider $T \div x_i y_{i+1}$ and note we may now regard $x_i y_{i+1}$ as a black edge because it was white in T itself. There must be a white neighbor of x_i in the realization and that must be $y_j \in Y_j$ for some $j > i + 1$. But y_j is a white neighbor of y_{i+1} also. So, $x_i y_j y_{i+1}$ is a realization of $\overline{P_3}$. Suppose there is an additional vertex, w , in the realization of the P_4 . If $w \in Y - Y_j$, then $w y_{i+1} y_j$ induces a white triangle and if $w \in Y_j$ then $w x_i y_j y_{i+1}$ has a white C_4 . Both of these cases produce a contradiction. If $w \in X$, then $w x_i$ is black and being in the realization of P_4 requires $w y_j$ to be black. However, this means $w y_{i+1}$ is also black because $j > i + 1$. This is also a contradiction. Hence there is no realization of P_4 in $T \div x_i y_{i+1}$.

This proves Claim 4.14. □

With Claim 4.15, the proof of the lemma is almost finished, leaving only two exceptional cases.

Claim 4.15. *There are at least $\left\lceil \frac{|X|+|Y|}{3} \right\rceil$ gray edges unless one of the following cases occurs:*

- (i.) $|Y_1| = \dots = |Y_{\ell+1}| = 1$ and $|X_1|, \dots, |X_\ell| \geq 2$.
(ii.) $|X_1| = \dots = |X_\ell| = 1$ and $|Y_2|, \dots, |Y_\ell| \geq 2$ but $Y_1 = Y_{\ell+1} = \emptyset$.

Proof of Claim 4.15. First suppose $Y_1 \neq \emptyset$. Consider the pairs (X_i, Y_i) for $i = 1, \dots, \ell$. By Claim 4.14, at least one of the sets must have size at least 2. If both have size at least 2, then the number of gray edges in $X_i \cup Y_i$ is at least $\left\lceil \frac{|X_i|+1}{3} \right\rceil + \left\lceil \frac{|Y_i|+1}{3} \right\rceil \geq \left\lceil \frac{|X_i|+|Y_i|+2}{3} \right\rceil$. If not, say $|Y_i| \geq 2$, then the number of gray edges in $X_i \cup Y_i$ is at least $\left\lceil \frac{|Y_i|+1}{3} \right\rceil = \left\lceil \frac{|X_i|+|Y_i|}{3} \right\rceil$. So, in this case, the total number of gray edges in $X \cup Y$ is at least $\left\lceil \frac{|X|+|Y|}{3} \right\rceil$ unless $|Y_{\ell+1}| = 1$ and there are ℓ other components of size 1. By Claim 4.14, this can only occur if $|Y_1| = \dots = |Y_\ell| = 1$. This is case (i.).

Second, suppose $Y_1 = \emptyset$. Consider the pairs (X_i, Y_{i+1}) for $i = 1, \dots, \ell$. By Claim 4.14, at least one of the sets must have size at least 2. If both have size at least 2, then the number of gray edges in $X_i \cup Y_{i+1}$ is at least $\left\lceil \frac{|X_i|+1}{3} \right\rceil + \left\lceil \frac{|Y_{i+1}|+1}{3} \right\rceil \geq \left\lceil \frac{|X_i|+|Y_{i+1}|+2}{3} \right\rceil$. If not, say $|Y_{i+1}| \geq 2$, then the number of gray edges in $X_i \cup Y_{i+1}$ is at least $\left\lceil \frac{|Y_{i+1}|+1}{3} \right\rceil = \left\lceil \frac{|X_i|+|Y_{i+1}|}{3} \right\rceil$. So, in this case, the total number of gray edges in $X \cup Y$ is at least $\left\lceil \frac{|X|+|Y|}{3} \right\rceil$ unless $|X_1| = 1$ and there are $\ell - 1$ other components of size 1. By Claim 4.14, this can only occur if $|X_2| = \dots = |X_\ell| = 1$. This is case (ii.).

Therefore, the only cases that remain are case (i.) and (ii.), completing the proof of Claim 4.15. \square

Case (i.) Consider the trigraph, T' , induced by $V(T) - (X_\ell \cup Y_{\ell+1})$. We claim that T' is P_4 -induced-saturated. Suppose not and consider $T \vdash e$ such that e is a white or black edge with both endpoints in $V(T')$. No realization of P_4 in T can have exactly three of its vertices in $V(T')$ because the pair $(Y_{\ell+1}, V(T'))$ has only white edges, giving a vertex of degree 0, and $(X_\ell, V(T'))$ has only black edges, giving a vertex of degree 3. No realization of P_4 in T can have two vertices in $V(T')$ and two in X_ℓ , which would give a C_4 . Finally, the realization cannot have $x_\ell \in X_\ell$ and the vertex in $Y_{\ell+1}$, making x_ℓ of degree 3. Therefore, the realization must have all four vertices in $V(T')$.

Hence, $V(T')$ is a P_4 -induced-saturated trigraph on at least 2 vertices and by the inductive hypothesis, the number of gray edges in T is at least

$$\left\lceil \frac{|V(T')|+1}{3} \right\rceil + \left\lceil \frac{|X_\ell|+1}{3} \right\rceil \geq \left\lceil \frac{|V(T')|+|X_\ell|+2}{3} \right\rceil = \left\lceil \frac{n+1}{3} \right\rceil.$$

Case (ii.) Consider the trigraph, T'' , induced by $V(T) - (X_\ell \cup Y_\ell)$. We claim that T'' is P_4 -induced-saturated. Suppose not and consider $T \vdash e$ such that e is a white or black edge with both endpoints in $V(T'')$. No realization of P_4 in T can have exactly three of its vertices in $V(T'')$ because the pair $(X_\ell, V(T''))$ has only black edges, giving a vertex of degree 3, and $(Y_\ell, V(T''))$ has only white edges, giving a vertex of degree 0. No realization of P_4 in T can have two vertices in $V(T'')$ and two in Y_ℓ , which would give a $\overline{C_4}$. Finally, the realization cannot have a vertex in Y_ℓ and the vertex x_ℓ in X_ℓ , making x_ℓ of degree 3. Therefore, the realization must have all four vertices in $V(T'')$.

Hence, $V(T'')$ is a P_4 -induced-saturated trigraph on at least 2 vertices and by the inductive hypothesis, the number of gray edges in T is at least

$$\left\lceil \frac{|V(T'')|+1}{3} \right\rceil + \left\lceil \frac{|Y_\ell|+1}{3} \right\rceil \geq \left\lceil \frac{|V(T'')|+|Y_\ell|+2}{3} \right\rceil = \left\lceil \frac{n+1}{3} \right\rceil.$$

This concludes the proof of Lemma 4.11. \square

5. PROOFS OF FACTS

In this section we will provide proofs to the general facts stated earlier.

Proof of Fact 4.2. Let T be a P_4 -induced-saturated trigraph. We want to show that its complement is also P_4 -induced-saturated. If \overline{R} is a realization of \overline{T} with an induced P_4 , then R is a realization of T with an induced P_4 . Thus, \overline{T} has no realization with an induced P_4 . Similarly, if R is a realization of $T \div uv$ with an induced P_4 , then \overline{R} is a realization of $\overline{T} \div uv$ with an induced P_4 . So, \overline{T} must be P_4 -induced-saturated. This proves Fact 4.2. \square

Proof of Fact 4.3. Using Fact 4.2, and without loss of generality, we may assume that all edges in $T[V_1, V_2]$ are white. We only need to show that $T[V_1]$ is P_4 -induced-saturated, $T[V_2]$ follows identically. If $T[V_1]$ is a gray complete graph or a single vertex, we are finished because a single vertex is trivially P_4 -induced-saturated. So let us assume that $T[V_1]$ is not a gray complete graph. Let e be a black or white edge in $T[V_1]$. Now, consider P , a realization of an induced P_4 in $T \div e$. As there are only white edges between V_1 and V_2 , it must be that no vertex of P can be in V_2 . So, $T[V_1]$ is P_4 -induced-saturated. This proves Fact 4.3. \square

Proof of Fact 4.4. Let none of the edges in $T[S, V - S]$ be gray. Let s_1s_2 be a black or white edge in S . Fact 4.2 allows us to assume that, without loss of generality, s_1s_2 is white. Consider P , a realization of an induced P_4 in $T \div s_1s_2$. We will show that P can not contain any vertices from $V - S$.

We consider two cases. First, suppose P has exactly one $v \in V - S$ and the fourth vertex is $s_3 \in S$. Since every pair of vertices in S has the same neighborhood in $V - S$, either vs_i is white for $i = 1, 2, 3$ or vs_i is black for $i = 1, 2, 3$. This gives a vertex in P either with degree 0 or degree 3, hence P cannot be an induced P_4 .

Second, suppose P has exactly two vertices $v_1, v_2 \in V - S$. Since the neighborhoods of s_1 and s_2 are the same in $V - S$, either (1) s_iv_j is black for $i, j \in \{1, 2\}$ or (2) s_iv_j is white for $i, j \in \{1, 2\}$ or, without loss of generality, (3) s_iv_1 is black for $i \in \{1, 2\}$ and s_iv_2 is white for $i \in \{1, 2\}$.

If (1) occurs, then P has a C_4 , a contradiction. If (2) occurs, then P has a $\overline{C_4}$, a contradiction. If (3) occurs, then either v_1 has degree 3 in P or v_2 has degree 0 in P . In any case, we have a contradiction. Thus, P must have all 4 vertices in S and so $T[S]$ is P_4 -induced-saturated. This proves Fact 4.4. \square

Proof of Fact 4.5. This fact is equivalent to saying T cannot have a gray P_4 . For the sake of contradiction, suppose that it does. By Fact 4.2, we may assume, without loss of generality, that there are 4 vertices that form a gray P_4 and have either zero black edges or one black edge. Let $abcd$ be the gray P_4 .

If there are no black edges in $T[\{a, b, c, d\}]$, then $abcd$ is a realization of P_4 . If ac is the sole black edge, then $bacd$ is a realization of P_4 . If bd is the sole black edge, then $abdc$ is a realization of P_4 . Finally, if ad is the sole black edge, then $badc$ is a realization of P_4 .

In all cases, there is a realization of P_4 in T , a contradiction to T being P_4 -induced-saturated. This proves Fact 4.5. \square

Proof of Fact 4.6. Let T be a trigraph which is P_4 -induced-saturated such that $C = \{u, v_1, \dots, v_{k-1}\}$ is a gray star with center u .

First, we will show that the edges in $T[\{v_1, \dots, v_{k-1}\}]$ (i.e., the edges between the leaves) have the same color. If $k \geq 4$ and v_1v_2, v_1v_3 are white and v_2v_3 is black, then $v_1uv_2v_3$ is a realization of P_4 , a contradiction. See Figure 6. If $k \geq 4$ and v_1v_2, v_1v_3 are black and v_2v_3 is white, then $uv_3v_1v_2$ is a realization of P_4 , a contradiction. Since v_1, v_2, v_3 could be chosen arbitrarily, $T[\{v_1, \dots, v_{k-1}\}]$ is monochromatic.

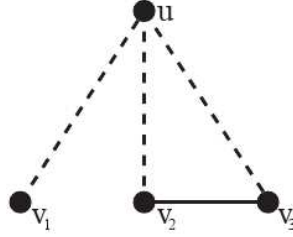


FIGURE 6. An example for the proof of Fact 4.6. A gray star with induced P_4 namely $v_1uv_2v_3$.

Second, we will show that, for every vertex $w \in V(T) - C$, the edges wv_i have the same color. If $k \geq 3$ and wv_1 is black and wv_2 is white, then either wv_1uv_2 or wv_1v_2u is a realization of P_4 , depending on the color of v_1v_2 . Since v_1, v_2 could be chosen arbitrarily, $T[\{w\}, C - \{u\}]$ is monochromatic for any w .

Let $X = \{x \in V(T) - C : T[\{x\}, C] \text{ is black}\}$, let $Y = \{y \in V(T) - C : T[\{y\}, C] \text{ is white}\}$ and let $Z = V(T) - (X \cup Y \cup C)$. For each $z \in Z$, either both the edge zu is white and $T[\{z\}, C - \{u\}]$ is black or both the edge zu is black and $T[\{z\}, C - \{u\}]$ is white.

It remains to show that the vertices in Z behave as set forth in the fact. If $k \geq 3$, we may assume, without loss of generality, that the edges of $T[\{v_1, \dots, v_{k-1}\}]$ are white. If $z \in Z$ has the property that zu is white and $T[\{z\}, C - \{u\}]$ is black, then v_1zv_2u is a realization of P_4 , a contradiction. Thus, all $z \in Z$ must have the property that $T[\{z\}, C - \{u\}]$ has the same color as $T[\{v_1, \dots, v_{k-1}\}]$ and zu is the complementary color. This is exactly the condition given in the statement of the fact.

If $k = 2$, then we just need to verify that $z_1, z_2 \in Z$ have the same neighborhood in $\{u, v_1\}$. If they do not, then we may assume that z_1u and z_2v_1 are black and z_1v_1 and z_2u are white. In this case, either $z_1uv_1z_2$ or $uz_1z_2v_1$ is a realization of P_4 depending on the color of z_1z_2 .

This classifies all the vertices in $V(T)$ and proves Fact 4.6. \square

Proof of Fact 4.7. Let T be a trigraph which is P_4 -induced-saturated such that $T[\{v_1, v_2, v_3\}]$ is a gray triangle. If Fact 4.7 fails to hold, then there exists an $x \in V(T) - \{v_1, v_2, v_3\}$ such that, without loss of generality, v_1x is black and v_2x, v_3x are white. But in this case, T has a realization of P_4 , namely $xv_1v_2v_3$, as is seen in Figure 7. This proves Fact 4.7. \square

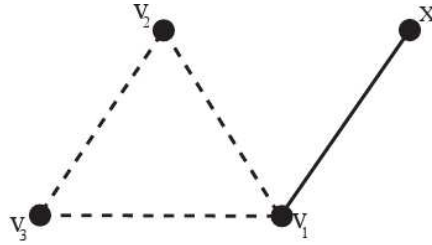


FIGURE 7. Example for proof of Fact 4.7. T contains a realization of P_4 , namely $xv_1v_2v_3$.

6. CONCLUSION

It is not clear whether our constructions of n -vertex P_4 -induced-saturated trigraphs with $\lceil (n+1)/3 \rceil$ gray edges are unique (up to complementation) for some cases of n . For instance, suppose n is divisible by 3 and T is constructed as follows: $V(T) = Z \cup C_0 \cup \{y\}$ where $C_0 = \{u, v_1, v_2\}$ induces a gray star on 3 vertices with a black edge v_1v_2 , the edges in $T[Z, \{v_1, v_2\}]$ are black, $T[Z]$ is a P_4 -induced-saturated

trigraph on $n - 4$ vertices with $\lceil (n - 3)/3 \rceil$ gray edges and all other edges in T are white. This T has $\lceil (n + 3)/3 \rceil = \lceil (n + 1)/3 \rceil$ gray edges.

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